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Stability of Algorithms for Waves with Large Flows

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We have identified a numerical instability that appears in algorithms for the linear propagation of waves in the presence of an advective flow. This instability is due to the coupling between the advective and wave terms and cannot be identified if stability conditions are derived separately for these two terms. It can appear in explicit or semi-implicit calculations using upwinded or centered spatial differences. We show that a stable scheme can be obtained by introducing a predictor step for the wave terms. When the semi-implicit treatment of the waves is used, the semi-implicit operator must be applied in the predictor step as well as in the corrector step. We present an improved formulation of the semi-implicit coefficient to take advection into account. © 1999 Academic Press

1. INTRODUCTION

The magnetohydrodynamic (MHD) equations are often used to study the low-frequency, long-wavelength behavior of plasmas. Strongly magnetized, slowly flowing plasmas are frequently encountered both in the laboratory and in astrophysics. In this situation, time integration of the MHD equations by explicit methods can be very inefficient for following the evolution of the plasma. The explicit treatment of the waves requires very small time steps because of the very large Alfvén speed present in the plasma. In order to obtain solutions efficiently, semi-implicit schemes for MHD have been developed [1–8]. A semi-implicit algorithm is more efficient than a fully implicit one, but the method still allows the time step to be chosen according to considerations of accuracy rather than stability [3]. The only time step restriction comes from the explicit treatment of the advective terms. The stability analysis of these algorithms has typically been carried out by considering wave propagation and advection separately [3, 4, 8].

We have extended these algorithms to study the global structure and dynamics of the solar corona [9–11] For this case one must confront the wide range of plasma parameters

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spanned by the solar atmosphere. Near the solar surface, plasma motions are slow, and the hot, strongly magnetized plasma is both subsonic and sub-Alfvénic. An implicit treatment of the waves is again necessary for an economical calculation. However, a few solar radii above the solar surface, advection dominates. Here the coronal plasma expands outward as the solar wind becomes both supersonic and super-Alfvénic. We have found that when the flow speed is large, traditional methods [3, 4, 8] for the advancement of the MHD equations can fail because of a linear numerical instability. This instability was not detected in the past either because the flow speed was not large enough to trigger the instability or because the viscosity and resistivity were large enough to stabilize the algorithm.

In this paper we discuss how the coupling of the wave-like terms with the advective terms may introduce a numerical instability. We show how this instability can be suppressed and how even the stability of explicit schemes can be improved. This instability cannot be identified if stability conditions are derived separately for wave propagation and for advection. A heuristic method developed by Hirt [12] is applied to investigate the instability. The proposed improvements involve the use of new predictor steps that include a fraction of the wave terms. An algorithm that did not suffer from this instability was presented by Lerbinger and Luciani [6].

We also discuss the stability of the semi-implicit algorithm, and we generalize the choice of the semi-implicit coefficient for the case when advection is significant. In the Appendix we briefly discuss how stable second-order accurate algorithms can be implemented.

2. THE WAVE–ADVECTION INSTABILITY

The phenomenon we are investigating can be illustrated with the help of a simplified subset of the MHD equations. Let us consider the following system of linear partial differential equations:

$$
\begin{aligned}
\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} &= -c\frac{\partial b}{\partial x}, \\
\frac{\partial b}{\partial t} + v\frac{\partial b}{\partial x} &= -c\frac{\partial a}{\partial x}.
\end{aligned}
\tag{1}
$$

This pair of equations describes the linear propagation of waves in one dimension, in the presence of a fixed uniform advective flow v. For example, in the case of a sound wave, a would be proportional to the perturbed pressure, b would be proportional to the perturbed velocity, and c would be the sound speed. The dispersion relation for Eq. (1) for waves of the form $e^{i(kx-\omega t)}$ is

$$
\omega = k(v \pm c). \tag{2}
$$

We will show that the coupling between advection and waves may cause a numerical instability in a finite-difference implementation of (1), how the instability can be avoided, and how the stable algorithm needs to be modified when we introduce a semi-implicit operator. To investigate the stability we will use the heuristic method described by Hirt [12], which consists of reducing finite-difference equations to differential equations by expanding terms in a Taylor series. The zero-order term represents the original differential equation, whereas higher-order terms (truncation errors) determine the stability properties.

Hirt's method applies in the small wave number limit and gives necessary conditions for stability (see [15], where its relation with the usual Von Neumann stability analysis is pointed out).

2.1. *Centered Predictor–Corrector*

The following algorithm represents one numerical discretization of Eq. (1). It is based on a leapfrog formulation of the wave terms in time, and a centered in space predictor–corrector formulation for the advective terms,

$$\frac{a_j^* - a_j^{n-1/2}}{\Delta t} + v\frac{a_{j+1}^{n-1/2} - a_{j-1}^{n-1/2}}{2\Delta x} = -\beta c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \qquad \text{P} \tag{3}$$

$$\frac{a_j^{n+1/2} - a_j^{n-1/2}}{\Delta t} + v\frac{a_{j+1}^* - a_{j-1}^*}{2\Delta x} = -c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \qquad \text{C} \tag{4}$$

$$\frac{b_{j+1/2}^* - b_{j+1/2}^n}{\Delta t} + v\frac{b_{j+3/2}^n - b_{j-1/2}^n}{2\Delta x} = -\beta c\frac{a_{j+1}^{n+1/2} - a_j^{n+1/2}}{\Delta x}, \qquad \text{P} \tag{5}$$

$$\frac{b_{j+1/2}^{n+1} - b_{j+1/2}^n}{\Delta t} + v\frac{b_{j+3/2}^* - b_{j-1/2}^*}{2\Delta x} = -c\frac{a_{j+1}^{n+1/2} - a_j^{n+1/2}}{\Delta x}, \qquad \text{C} \tag{6}$$

where "P" and "C" indicate respectively the predictor and the corrector steps. Both a and b are defined on meshes that are staggered in space and time: $a_j^{n+1/2} \equiv a(n\Delta t + \Delta t/2, j\Delta x)$, $b_{j+1/2}^n \equiv b(n\Delta t, j\Delta x + \Delta x/2)$.

The choice of staggered meshes (in space and time) is motivated by the leapfrog advance of the wave terms. A straightforward analysis of the predictor–corrector treatment of the scalar advective equation

$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = 0 \tag{7}$$

shows that this algorithm for advection is stable when an appropriate CFL condition on the time step is satisfied [8]. Traditionally, these two algorithms were combined (with $\beta = 0$) when both advection and waves were present simultaneously [3, 4, 8]. We have recently found that the coupling between advection and the wave terms creates an instability. We will show below that the term involving β in the predictor is required to produce a stable algorithm. It is precisely the cure of this instability that is a central contribution of this paper.

We expand all quantities in Eqs. (3)–(6) about $(n\Delta t, j\Delta x)$ and omit the indices. This gives us the equations

$$\begin{aligned}
\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} + c\frac{\partial b}{\partial x} &= \frac{v^2\Delta t}{2}\frac{\partial^2 a}{\partial x^2} - vc\Delta t\left(\frac{1}{2} - \beta\right)\frac{\partial^2 b}{\partial x^2} + O(\Delta^2), \\
\frac{\partial b}{\partial t} + v\frac{\partial b}{\partial x} + c\frac{\partial a}{\partial x} &= \frac{v^2\Delta t}{2}\frac{\partial^2 b}{\partial x^2} - vc\Delta t\left(\frac{1}{2} - \beta\right)\frac{\partial^2 a}{\partial x^2} + O(\Delta^2),
\end{aligned} \tag{8}$$

where we have eliminated mixed derivatives using Eq. (1). For convenience, we assume that $v \geq 0$. This is not a restrictive assumption, since an analysis with negative v yields the same results. The dispersion relation for this system is

$$\omega = k(v \pm c) - i\frac{v^2 \Delta t}{2}k^2 \pm i vc\Delta t\left(\frac{1}{2} - \beta\right)k^2. \tag{9}$$

Note that in the limit $\Delta t \to 0$ we retrieve the correct dispersion relation in Eq. (2), as expected. The scheme is unstable if the imaginary part of ω is positive. The second term on the right-hand side of Eq. (9) is a damping term, which is due to the predictor–corrector scheme. The third term, which involves vc, is due to the coupling of the waves and advection, and may cause instability (when $v < c$) unless $\beta = \frac{1}{2}$ [13]. The finite Δx and Δt dispersion relation for this algorithm is given in Section 3.1. In particular, previous algorithms [3, 4, 8] (which had $\beta = 0$), were unstable to this mode; these algorithms were only physically useful when the dissipation exceeded the growth rate of this mode. Since the growth rate of the instability is proportional to v, we did not discover this mode until we simulated problems with significant flow. This is the parameter regime for our coronal simulation studies, in which the solar wind expands supersonically and super-Alfvénically away from the Sun.

This advection algorithm is only first-order accurate in time. In the Appendix we briefly discuss the second-order accurate Adams–Bashforth/Adams–Moulton predictor–corrector. Since we are interested in problems with shocks and discontinuities the utility of second-order advection schemes may be limited. As is well known [14, p. 345], higher-order advection schemes are not monotonic near steep gradients.

2.2. *Upwinding and Predictor–Corrector*

A nonlinear instability may appear in the centered predictor–corrector scheme when steep profiles are present. For this reason upwinding is used as an alternative. The wave–advection coupling influences the stability of this algorithm too. We write the finite-difference algorithm as

$$\frac{a_j^{n+1/2} - a_j^{n-1/2}}{\Delta t} + v\frac{a_j^{n-1/2} - a_{j-1}^{n-1/2}}{\Delta x} = -c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \tag{10}$$

$$\frac{b_{j+1/2}^{n+1} - b_{j+1/2}^n}{\Delta t} + v\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x} = -c\frac{a_{j+1}^{n+1/2} - a_j^{n+1/2}}{\Delta x}, \tag{11}$$

where we have assumed that $v \geq 0$. Notice that no predictor step is present. Application of Hirt's method [12] gives

$$\begin{aligned}\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} + c\frac{\partial b}{\partial x} &= \left(\frac{v\Delta x}{2} - \frac{v^2\Delta t}{2}\right)\frac{\partial^2 a}{\partial x^2} + \frac{vc\Delta t}{2}\frac{\partial^2 b}{\partial x^2} + O(\Delta^2),\\ \frac{\partial b}{\partial t} + v\frac{\partial b}{\partial x} + c\frac{\partial a}{\partial x} &= \left(\frac{v\Delta x}{2} - \frac{v^2\Delta t}{2}\right)\frac{\partial^2 b}{\partial x^2} + \frac{vc\Delta t}{2}\frac{\partial^2 a}{\partial x^2} + O(\Delta^2).\end{aligned} \tag{12}$$

The corresponding dispersion relation is

$$\omega = k(v \pm c) - i\left(\frac{v\Delta x}{2} - \frac{v^2\Delta t}{2} \pm \frac{vc\Delta t}{2}\right)k^2. \tag{13}$$

Notice that a destabilizing vc term is present in this case as well. For numerical stability, a necessary condition is

$$(|v| + c)\frac{\Delta t}{\Delta x} < 1. \tag{14}$$

This is the well-known "CFL condition" for explicit stability, as in [14, p. 290]. For these equations the stability region can be enlarged by eliminating the vc term as in Section 2.1. For this purpose we have devised the following algorithm, which has predictor steps for the wave terms only:

$$\frac{a_j^* - a_j^{n-1/2}}{\Delta t} = -\frac{c}{2}\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \qquad \text{P} \tag{15}$$

$$\frac{a_j^{n+1/2} - a_j^{n-1/2}}{\Delta t} + v\frac{a_j^* - a_{j-1}^*}{\Delta x} = -c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \qquad \text{C} \tag{16}$$

$$\frac{b_{j+1/2}^* - b_{j+1/2}^n}{\Delta t} = -\frac{c}{2}\frac{a_{j+1}^{n+1/2} - a_j^{n+1/2}}{\Delta x}, \qquad \text{P} \tag{17}$$

$$\frac{b_{j+1/2}^{n+1} - b_{j+1/2}^n}{\Delta t} + v\frac{b_{j+1/2}^* - b_{j-1/2}^*}{\Delta x} = -c\frac{a_{j+1}^{n+1/2} - a_j^{n+1/2}}{\Delta x}. \qquad \text{C} \tag{18}$$

It is easy to verify that the wave–advection coupling terms disappear from Eq. (13), giving the following necessary condition for stability,

$$v\frac{\Delta t}{\Delta x} < 1. \tag{19}$$

Note that this scheme allows larger time steps to be taken than the scheme presented in Eqs. (10)–(11). For further discussion see Section 3.3. Of course, there is also the additional requirement that

$$c\frac{\Delta t}{\Delta x} < 1, \tag{20}$$

from the explicit leapfrog treatment of the wave terms (see, for example, [14, p. 260]). In the Appendix we show how to implement a second-order accurate upwind scheme that does not suffer from the wave–advection instability.

2.3. *Introducing the Semi-implicit Term*

The leapfrog algorithm for the advancement of the linear wave equation is restricted to small time steps as specified by Eq. (20). A fully implicit treatment of the wave terms can remove this restriction. However, when realistic multidimensional cases are considered, this may require the time-consuming inversion of nearly intractable matrices. The philosophy behind the semi-implicit algorithm, as applied to MHD, can be found in Refs. [1–3]. The semi-implicit method ensures stability through adispersive term added to Eq. (1), which we rewrite as

$$\begin{aligned} \frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} &= -c\frac{\partial b}{\partial x}, \\ \left(1 - C^2\Delta t^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial b}{\partial t} + v\frac{\partial b}{\partial x} &= -c\frac{\partial a}{\partial x}, \end{aligned} \tag{21}$$

where C^2 is the semi-implicit coefficient. This coefficient needs to be chosen for stability based on the value of Δt [3]. The advantage of the semi-implicit method over a fully implicit scheme is the introduction of symmetric operators that are more easily inverted. The relationship between fully implicit and semi-implicit schemes is outlined by Caramana [16]. For the scheme presented in Section 2.2, Eqs. (15)–(16) remain the same, but Eqs. (17)–(18) are modified as

$$(1-\beta_s\mathbf{D})\frac{b^*_{j+1/2}-b^n_{j+1/2}}{\Delta t} = -\frac{c}{2}\frac{a^{n+1/2}_{j+1}-a^{n+1/2}_{j}}{\Delta x}, \qquad \text{P} \qquad (22)$$

$$(1-\mathbf{D})\frac{b^{n+1}_{j+1/2}-b^n_{j+1/2}}{\Delta t} + v\frac{b^*_{j+1/2}-b^*_{j-1/2}}{\Delta x} = -c\frac{a^{n+1/2}_{j+1}-a^{n+1/2}_{j}}{\Delta x}, \qquad \text{C} \qquad (23)$$

where $\mathbf{D}$ represents the numerical implementation of the semi-implicit operator. β_s is a factor to be determined later for optimum stability. In the past [3, 4, 8] we had used $\beta_s = 0$ (i.e., no semi-implicit term in the predictor), which does not give maximum stability. We now show that $\beta_s = 1$ gives optimum stability. Applying Hirt's method, we can show that the unstable wave–advection coupling term is

$$\frac{vc\Delta t}{2}\left[(1-\mathbf{D})^{-1} - (1-\beta_s\mathbf{D})^{-1}\right]\frac{\partial^2 a}{\partial x^2}, \qquad (24)$$

which disappears if and only if $\beta_s = 1$.

3. VON NEUMANN STABILITY ANALYSIS

The results presented in the previous section can be incorporated into a single general algorithm, the stability of which will be examined in detail in this section. We present the algorithm and derive its dispersion relation. However, a simpler dispersion relation can be derived in a heuristic way. From that we can extract an expression for the semi-implicit coefficient. The correctness of our assumption is proved *a posteriori* through a numerical analysis of the full dispersion relation.

3.1. *The Full Dispersion Relation*

We write a general algorithm for wave propagation in the presence of advection as

$$\frac{a^*-a^{n-1/2}}{\Delta t} + \beta^a_f v\frac{\Delta a}{\Delta x}^{n-1/2} = -\beta^a_w c\frac{\Delta b^n}{\Delta x}, \qquad \text{P} \qquad (25)$$

$$\frac{a^{n+1/2}-a^{n-1/2}}{\Delta t} + v\frac{\Delta a^*}{\Delta x} = -c\frac{\Delta b^n}{\Delta x}, \qquad \text{C} \qquad (26)$$

$$(1-\beta_s\mathbf{D})\,\frac{b^*-b^n}{\Delta t} + \beta^b_f v\frac{\Delta b^n}{\Delta x} = -\beta^b_w c\frac{\Delta a}{\Delta x}^{n+1/2}, \qquad \text{P} \qquad (27)$$

$$(1-\mathbf{D})\,\frac{b^{n+1}-b^n}{\Delta t} + v\frac{\Delta b^*}{\Delta x} = -c\frac{\Delta a}{\Delta x}^{n+1/2}, \qquad \text{C} \qquad (28)$$

where we have simplified the notation. In order to be as general as possible, we have introduced several nonnegative numerical factors: β^a_f and β^b_f set the fraction of the advective term in the predictors of the equation for a and b; β^a_w and β^b_w do the same for the wave term; β_s sets the fraction of the semi-implicit term in the predictor equation for b. In order to perform a Von Neumann stability analysis [14, p. 70], we introduce the following terms in (25)–(28),

$$a_j^{n+1/2} = az^{n+1/2}e^{ikj\Delta x}, \tag{29}$$

$$b_{j+1/2}^n = bz^n e^{ik(j+1/2)\Delta x}, \tag{30}$$

where $z = e^{-i\omega\Delta t}$. After simplification, we obtain the dispersion relation

$$\begin{aligned} &w^2 S^{(2)} + w\left[(C_w Q_w)^2 Q_a^{(2)} Q_b^{(2)} + C_f Q_f\left(S^{(2)} Q_a^{(1)} + Q_b^{(1)}\right)\right] \\ &\quad + (C_w Q_w)^2 Q_a^{(2)} Q_b^{(2)} + (C_f Q_f)^2 Q_a^{(1)} Q_b^{(1)} = 0, \end{aligned} \tag{31}$$

where

$$\begin{aligned} w &= z - 1, \\ Q_w &= 2\sin\left(\frac{k\Delta x}{2}\right), \\ Q_f &= \begin{cases} 1 - e^{ik\Delta x} & \text{for upwinded differencing of advective terms} \\ i\sin(k\Delta x) & \text{for centered differencing of advective terms,} \end{cases} \\ S^{(1)} &= 1 + \beta_s C^2 \frac{\Delta t^2}{\Delta x^2} Q_w^2, \\ S^{(2)} &= 1 + C^2 \frac{\Delta t^2}{\Delta x^2} Q_w^2, \\ C_w &= \frac{c\Delta t}{\Delta x}, \\ C_f &= \frac{|v|\Delta t}{\Delta x}, \\ Q_a^{(1)} &= 1 - \beta_f^a C_f Q_f, \\ Q_a^{(2)} &= 1 - \beta_w^a C_f Q_f, \\ Q_b^{(1)} &= 1 - \beta_f^b \frac{C_f Q_f}{S^{(1)}}, \\ Q_b^{(2)} &= 1 - \beta_w^b \frac{C_f Q_f}{S^{(1)}}. \end{aligned}$$

For the algorithm to be numerically stable we need $|z| \leq 1$. The quadratic equation (31) can be solved analytically to investigate in detail the consequences of various choices of parameters (β_f^a, β_f^b, β_w^a , β_w^b, β_s, and the semi-implicit coefficient C^2) on the stability of the algorithm. Notice that in the limit $k\Delta x \to 0$ the stability limits derived from Eq. (31) are consistent with those derived from Eqs. (9) and (13). In particular, in previous work we developed a semi-implicit term for the case when flows were small, compared to the Alfvén and sound speeds [3]. In the application of our algorithm to the modeling of the solar wind, this assumption is not appropriate: the flow speed in fact exceeds the wave speeds, since the solar wind becomes supersonic and super-Alfvénic as it expands into interplanetary space. We have extended the formulation of the semi-implicit term to this case, as shown below. Extracting from Eq. (31) the analytical expression for the semi-implicit coefficient

C^2 that gives the maximum region of stability in (C_w, C_f) space (CFL numbers for waves and flow) is a formidable task. Hence we prefer to proceed in a more heuristic way.

3.2. *Heuristic Derivation of the Semi-implicit Coefficient*

In order to obtain an expression for the semi-implicit factor C^2 in the presence of advection, let us first consider the dispersion relation when $v = 0$. From Eq. (31), we can show that in this case,

$$\tilde{\omega}^2 = \frac{c^2\tilde{k}^2}{1 + C^2\tilde{k}^2\Delta t^2}, \tag{32}$$

where

$$\begin{aligned} \tilde{\omega} &= \sin\left(\frac{\omega\Delta t}{2}\right)\frac{2}{\Delta t}, \\ \tilde{k} &= \sin\left(\frac{k\Delta x}{2}\right)\frac{2}{\Delta x}. \end{aligned} \tag{33}$$

For stability ω must be real, which requires $\tilde{\omega} \le 2/\Delta t$. This gives the familiar expression for C^2 found in [3]:

$$C^2 \ge \max\left\{\frac{\Delta x^2}{4\Delta t^2}\left(C_w^2 - 1\right), 0\right\}. \tag{34}$$

Comparing Eq. (2) (the analytical dispersion relation when advection is present) with Eq. (32), we suggest the following ansatz to include the effect of advection in a simplified dispersion relation,

$$(\tilde{\omega} - \tilde{k}v)^2 = \frac{c^2\tilde{k}^2}{1 + C^2\tilde{k}^2\Delta t^2}. \tag{35}$$

Considering the most unstable case, $\tilde{k} = \tilde{k}_{\max} = 2/\Delta x$, we obtain the following inequality that the semi-implicit coefficient must satisfy:

$$C^2 \ge \max\left\{\frac{\Delta x^2}{4\Delta t^2}\left[\frac{C_w^2}{(1 - C_f)^2} - 1\right], 0\right\}. \tag{36}$$

Detailed analysis of the dispersion relation, Eq. (31), confirms that this choice of semi-implicit term indeed gives a stable algorithm even in the presence of significant flows (see Section 3.4). Note that when $C_f = 1$ (the CFL limit for pure advection), C^2 becomes infinite. A possible interpretation of this is that the semi-implicit coefficient cannot stabilize advection. This implies also that C_f must not be too close to one in order to limit the amount of artificial inertia introduced by the semi-implicit operator.

When this algorithm is implemented for the MHD equations, the semi-implicit coefficient is given by Eq. (36) using the local wave and flow speeds. We find that this formulation is stable and makes C^2 nonuniform in space, which eliminates the artificial inertia associated with the semi-implicit term when C_w is small.

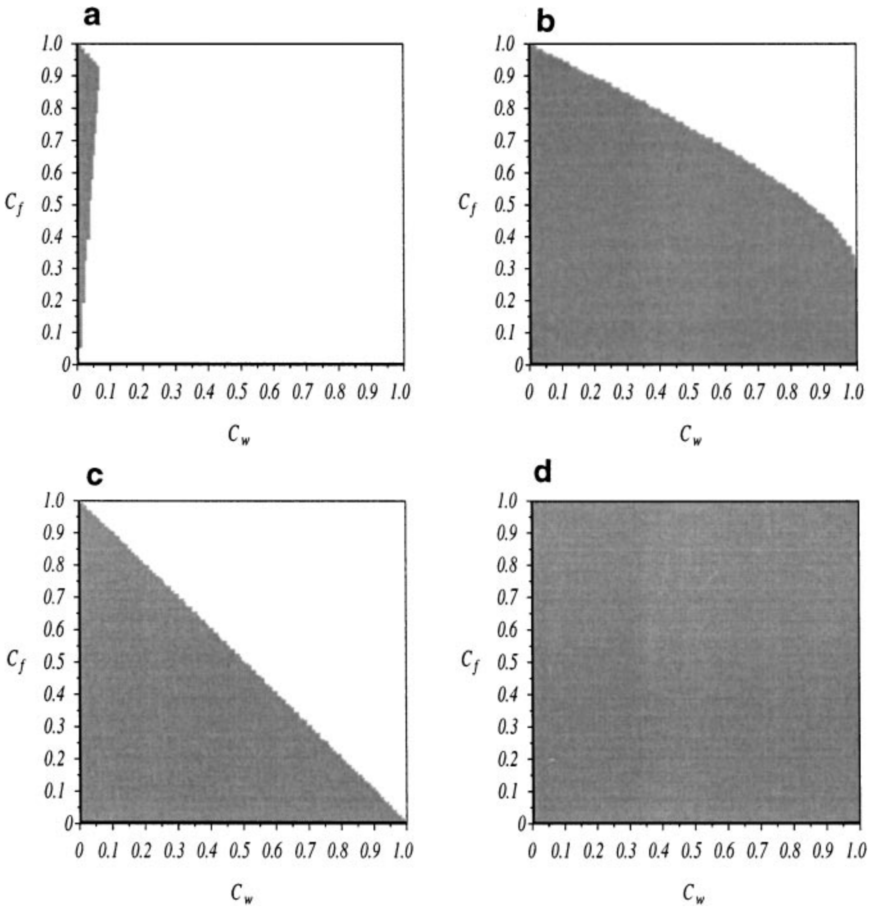

FIG. 1. Stability region in the (C_w, C_f) plane for explicit schemes ($C^2 = 0$): (a) centered differences, $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 0$; (b) centered differences, $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$; (c) upwind differences, $\beta_f^a = \beta_f^b = \beta_w^a = \beta_w^b = 0$; (d) upwind differences, $\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 1/2$. Note that the algorithms are stable for $C_f = 0$ and $C_w < 1$.

3.3. *Explicit Algorithms*

Equation (31) is a quadratic in w and can be solved exactly. We will study the solutions in the two parameter space (C_w, C_f). We consider explicit cases ($C^2 = 0$) first. Results are reported in Fig. 1, where the shaded area implies stability (i.e., if Eq. (31) predicts stability for all the possible wave numbers k). We first show that the traditional treatment of Eq. (1) (in which there are no wave terms in the predictor) has regions of instability. Namely, the stability of the centered predictor–corrector algorithm ($\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 0$) is presented in Fig. 1a, which shows that the scheme is explicitly stable when the flow is absent ($C_f = 0$) or there are no waves ($C_w = 0$). When the wave speed is small, the flow has a stabilizing effect due to the diffusive term on the right-hand sides of Eq. (8) that is proportional to $v^2 \Delta t/2$. Introducing the fractional wave terms in the predictor ($\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$) greatly improves the stability of the algorithm (see Fig. 1b), as expected from the discussion in Section 2.1.

Next, we address the scheme with upwinding. In Fig. 1c we present the stability plot for an upwind scheme without a predictor–corrector ($\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 0$). This result is in exact agreement with Eq. (14). When we introduce a predictor step containing a fraction of the wave terms ($\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 1/2$), we can extend the region of stability, as shown in Fig. 1d. We have tried to use different values for β_w^a and β_w^b, but the maximum stability region is found when both coefficients are one half in value, in accordance with the analysis in Section 2.

3.4. *Semi-Implicit Algorithms*

We now introduce a finite semi-implicit coefficient in the algorithm and study the stability for $C_w > 1$. Let us first examine a case with centered differences. We include wave terms in the predictor and the classic semi-implicit coefficient given by Eq. (34). The parameters are $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 0$. As shown in Fig. 2a, using a semi-implicit term only in the corrector cannot yield a stable algorithm for $C_w > 1$ unless there is no flow. Adding the semi-implicit term in the predictor (i.e., setting $\beta_s = 1$) greatly

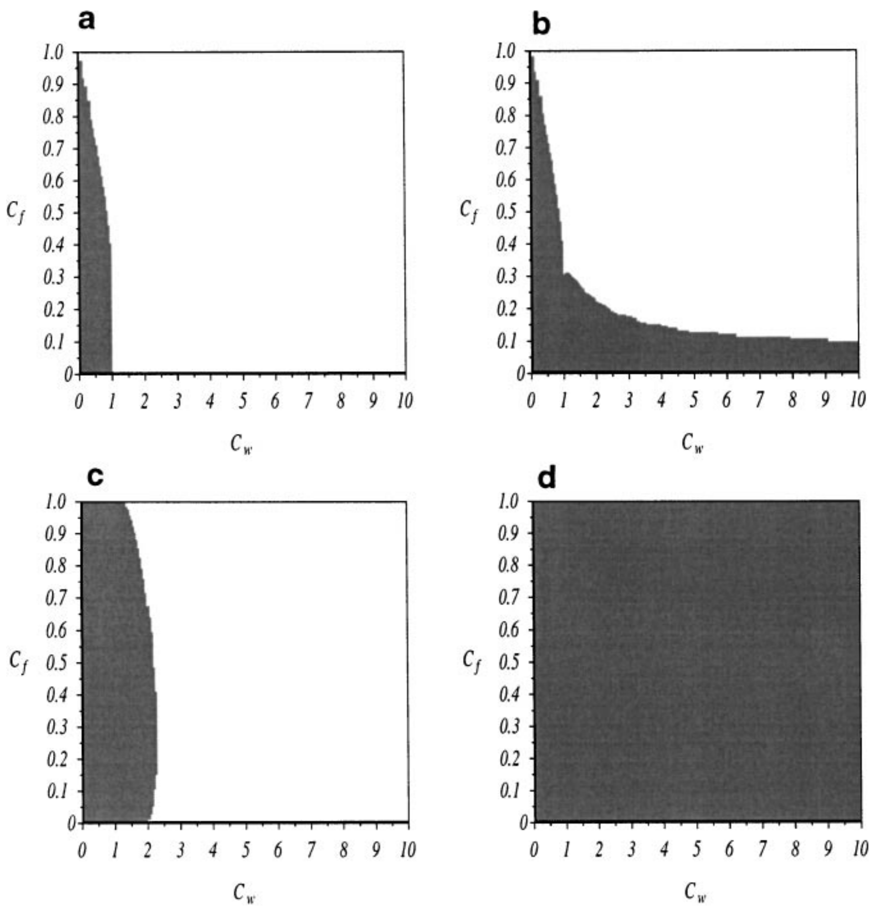

FIG. 2. Stability region in the (C_w, C_f) plane for semi-implicit schemes: (a) centered differences, classic semi-implicit coefficient, $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 0$; (b) centered differences, classic semi-implicit coefficient, $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 1$; (c) upwind differences, classic semi-implicit coefficient, $\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 0$; (d) full stability: centered differences, new semi-implicit coefficient, $\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 1$, or upwind differences, classic semi-implicit coefficient, $\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 1$. Note that the algorithms are stable for $C_f = 0$ and all C_w.

TABLE I
Parameters in Eqs. (25)–(28) That Characterize Algorithms Stable for $C_f < 1$ and any C_w

Differencing	SI coefficient	β_s	β_f^a	β_f^b	β_w^a	β_w^b
Centered	New	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Upwinded	Classic	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
Upwinded	New	1	$\leq\frac{1}{2}$	$\leq\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

improves the stability properties (see Fig. 2b). The region of stability can be extended even further (i.e., stable for all C_w, for all values of $C_f < 1$) by using the new choice of semi-implicit coefficient in Eq. (36) with centered differences and a predictor–corrector scheme ($\beta_f^a = \beta_f^b = 1$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 1$), as shown in Fig. 2d.

The situation is similar when upwind differences are used. For example, when one uses a predictor step for the waves and the classic semi-implicit coefficient in the corrector only ($\beta_f^a = \beta_f^b = 0$, $\beta_w^a = \beta_w^b = 1/2$, $\beta_s = 0$), the algorithm is stable for $C_w \lesssim 2$, as shown in Fig. 2c. Introducing the semi-implicit term in the predictor ($\beta_s = 1$) gives full stability for $C_f < 1$, as shown in Fig. 2d. It is interesting to note that in this case the classic semi-implicit coefficient is sufficient for stability. Even though stability can be obtained in this case by using the classic semi-implicit term, in general it is advisable to use the improved semi-implicit coefficient, Eq. (36), for the following reason. When this algorithm is implemented for the MHD equations, it is frequently difficult to separate the "advective" terms from the "wave" terms. For example, consider Faraday's equation for ideal MHD ($\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{37}$$

The term $\nabla \times (\mathbf{v} \times \mathbf{B})$ has both an advective component, $-\mathbf{v} \cdot \nabla \mathbf{B}$, and a wave component (the other parts). Therefore, it would be difficult to implement the above algorithm [upwind differences with no predictor for the advective terms ($\beta_f^a = \beta_f^b = 0$) and a predictor for the wave terms ($\beta_w^a = \beta_w^b = 1/2$)]. We have found that it is necessary to use the new semi-implicit coefficient when β_f^a and β_f^b are nonzero. Full stability for $C_f < 1$ can be obtained when $\beta_f^a \leq 1/2$ and $\beta_f^b \leq 1/2$ and the new semi-implicit coefficient is used. In particular the algorithm with $\beta_f^a = \beta_f^b = \beta_w^a = \beta_w^b = 1/2$, with the new semi-implicit coefficient (in both the predictor and the corrector, $\beta_s = 1$), and with upwind differences, is stable for all C_w for $C_f < 1$. This algorithm is our choice for advancing the full MHD equations.

In Table I we summarize the parameters that give fully stable algorithms for $C_f < 1$. An algorithm which had the semi-implicit and the wave terms in the predictor was presented by Lerbinger and Luciani [6].

4. CONCLUSION

In this paper we have analyzed the numerical stability of algorithms for the advancement of wave–advection equations. Our application is the time integration of the MHD equations. Traditional algorithms were developed for applications where the flow speed was small compared to the Alfvén and sound speeds [3, 4, 8]. While applying our MHD model to the solar wind, which becomes supersonic and super-Alfvénic in interplanetary space, we discovered a numerical instability that occurs in regions where the flow speed is large. By analyzing the stability of our algorithm, as applied to a simplified model with waves and advection, using Hirt's technique [12] and a Von Neumann stability analysis, we determined that the source of the instability was the coupling between wave-like terms in the leapfrog advance and the advective terms.

We have presented techniques for fixing this instability, and we have proved that the resulting algorithm is stable. The implementation of this improved algorithm in our spherical MHD code has resulted in a more robust code (for an application see Linker *et al.* [17]). We have also presented an improved formulation of the semi-implicit coefficient that takes into account the advective terms.

APPENDIX: SECOND-ORDER ACCURATE SCHEMES

In the paper we have examined advection schemes that are only first-order accurate. We now briefly discuss how the stability properties are modified when second-order accurate schemes are used. The central scheme presented in Section 2.1 can be made second-order accurate by changing to Adams–Bashforth/Adams–Moulton predictor–corrector. We rewrite Eqs. (3) and (4) as

$$\frac{a_j^* - a_j^{n-1/2}}{\Delta t} = -v\left(\frac{3}{2}\frac{a_{j+1}^{n-1/2} - a_{j-1}^{n-1/2}}{2\Delta x} - \frac{1}{2}\frac{a_{j+1}^{n-3/2} - a_{j-1}^{n-3/2}}{2\Delta x}\right) - \beta c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \quad (38)$$

$$\frac{a_j^{n+1/2} - a_j^{n-1/2}}{\Delta t} = -v\left(\frac{1}{2}\frac{a_{j+1}^* - a_{j-1}^*}{2\Delta x} + \frac{1}{2}\frac{a_{j+1}^{n-1/2} - a_{j-1}^{n-1/2}}{2\Delta x}\right) - c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}. \quad (39)$$

Equations (5) and (6) can be rewritten analogously. After we apply Hirt's method the following first-order term appears,

$$-(1-\beta)\frac{vc\Delta t}{2}\frac{\partial^2 b}{\partial x^2}, \quad (40)$$

which can be eliminated if we choose $\beta = 1$ because the provisional value a^* is actually an approximation for $a^{n+1/2}$ and so a full predictor step for the waves is needed.

Similarly we can convert the upwind scheme in Section 2.2 to second order by introducing the modified Beam–Warming [18] algorithm

$$\frac{a_j^* - a_j^{n-1/2}}{\Delta t} = -\beta c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}, \quad (41)$$

$$\frac{a_j^{n+1/2} - a_j^{n-1/2}}{\Delta t} = v\frac{3a_j^* - 4a_{j-1}^* + a_{j-2}^*}{2\Delta x} + v^2\Delta t\frac{a_j^* - 2a_{j-1}^* + a_{j-2}^*}{2\Delta x^2} - c\frac{b_{j+1/2}^n - b_{j-1/2}^n}{\Delta x}. \quad (42)$$

The resulting first-order error term is

$$-\left(\frac{1}{2} - \beta\right)\frac{vc\Delta t}{2}\frac{\partial^2 b}{\partial x^2}. \quad (43)$$

In this case β must be $\frac{1}{2}$. In fact the a^* terms in the corrector are actually an approximation for a^n. The same applies to Eqs. (17) and (18).

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